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# De summatione innumerabilium progressionum

Leonhard Euler

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# DE SUMMATIONE INNUMERABILIVM PROGRESSIONVM.

*Auct. L. Eulero.*

I.

**Q**uae in praecedente differtatione de progressionibus transcendentibus earumque terminis generalibus tradidi, multo latius patent, quam videri possent; et inter alia quam plurima, ad quae accommodari possunt, eximius earum potest esse vsus in inueniendis summis innumerabilium progressionum. Quemadmodum enim in superiore differtatione innumerae progressionis ad terminos generales sunt reuocatae, quae communem algebram transcendunt; ita hic eandem methodum accommodabo ad terminos summatorios inueniendos progressionum, ad quas indefinite summandas communis algebra non sufficit.

§. 2. Progressio quaequam summari dicitur indefinite, si detur formula numerum indefinitum  $n$  continens, quae exponat summam tot terminorum illius progressionis, quot  $n$  comprehendit unitates, ita vt si ponatur v. gr.  $n=10$  ea formula exhibeat summam decem terminorum a primo numeratorum. Formula haec vocatur terminus summatorius illius progressionis, atque est simul

M 2

ter-

terminus generalis progressionis, cuius terminus quicumque aequatur summae tot terminorum illius progressionis, quot eius exponens in se continet unitates.

§. 3. Cum progressionibus quaeque exponantur terminis generalibus, quaestio de summendis progressionibus est haec, ut ex termino generali terminus summatorius inveniatur. Et quidem iam eo est peruentum, ut, quoties terminus generalis est functio rationalis ipsius indicis  $n$  exponentes sunt numeri integri affirmativi, semper terminus summatorius inueniri queat. Quando autem exponentes ipsius  $n$  sunt negativi, nisi excipiantur pauci casus, nemo adhuc terminos summatorios dedit. Ratio huius difficultatis est, quod tum termini summatorii plerumque algebraice exprimi nequeant, sed tales requirant formas, quae quadraturas in se contineant.

§. 4. Assumatur haec forma  $\int \frac{1-x^n}{1-x} dx$ , tamquam terminus generalis cuiusdam progressionis; quae scilicet integrata, ita ut fiat  $=0$  si  $x=0$  positoque  $x=1$  daret terminum ordine  $n$ . Progressio quae hoc modo ex ea formatur erit haec  $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ , etc. cuius ergo terminus generalis est formula assumpta  $\int \frac{1-x^n}{1-x} dx$

Series

Series vero haec inuenta summatoria est progres-  
sionis harmonicae  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$  etc. cuius termi-  
nus generalis est  $\frac{1}{n}$ . Quamobrem huius progres-  
sionis terminus summatorius erit  $\int \frac{1-x^n}{1-x} dx$ , qui  
illius est terminus generalis.

§. 5. Cum terminus generalis progressionis  $1,$   
 $1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3},$  etc. sit  $\int \frac{1-x^n}{1-x} dx$ . Poterit ex  
hoc ea progressio interpolari, seu quilibet termi-  
nus medius inueniri: vt si requiratur terminus, cu-  
ius index est  $\frac{1}{2}$ , oportebit integrari  $\frac{1-\sqrt{x}}{1-x} dx$  vel  
 $\frac{dx}{1+\sqrt{x}}$ , cuius integrale est  $2\sqrt{x}-2l(1+\sqrt{x})$  quod  
cum fiat  $=0$  si  $x=0$ , ponatur  $x=1$ , erit termi-  
nus ordine  $\frac{1}{2}=2-2l2$ . Deinde, quia generali-  
ter terminus ordine  $n+1$ , terminum ordine  $n$   
superat fractione  $\frac{1}{n+1}$ , erit terminus ordine  $1\frac{1}{2}=$   
 $2\frac{2}{3}-2\sqrt{2}$ , et terminus ordine  $2\frac{1}{2}$  hic  $2+\frac{2}{3}+\frac{2}{5}-$   
 $2l2$ , etc. Series igitur interpolata erit

$$2-2l2, 1, 2+\frac{2}{3}-2l2, 1+\frac{1}{2}, 2+\frac{2}{3}+\frac{2}{5}-2l2, \text{etc.}$$

§. 6. Ad hunc modum rem generalius com-  
plexus sum, et assumi formulam  $\int \frac{1-P^n}{1-P} dx$ , vbi  
P denotat functionem quamcunque ipsius  $x$ . In-  
tegrale hoc vt semper ita debet accipi vt posi-  
to  $x=0$ ; id totum fiat  $=0$ . Deinde hoc facto

non vt ante pono  $x=1$ , sed vt latius pateat pono  $x=k$ . Forma hoc modo resultans erit terminus ordine  $n$  progressionis cuiusdam, cuius terminus generalis est forma assumpta  $\int \frac{1-P^n}{1-P} dx$ . Progressio vero ipsa haec erit  $k, k+\int P dx, k+\int P dx + \int P^2 dx$ , etc. vbi in integralibus  $\int P dx, \int P^2 dx$ , etc. loco  $x$  iam positum esse  $k$  pono.

§. 7. Progressio inuenta, si quivis terminus a sequeute subtrahatur, praebebit hanc  $k, \int P dx, \int P^2 dx, \int P^3 dx$  etc. Huiusque terminus summatorius aequalis est termino generali praecedentis progressionis, cuius terminus generalis est  $\int P^{n-1} dx$ , haec formula  $\int \frac{1-P^n}{1-P} dx$ . Sit  $P=x^\alpha: a^\alpha$  erit progressio huius  $k, \frac{k^{\alpha+1}}{(\alpha+1)a^\alpha}, \frac{k^{2\alpha+1}}{(3\alpha+1)a^{2\alpha}}$  etc. terminus generalis  $\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}}$  atque terminus summatorius hic  $\int \frac{a^{n\alpha}-x^{n\alpha}}{(a^\alpha-x^\alpha)a^{n\alpha-\alpha}} dx$ .

§. 8. Inventus ergo est terminus summatorius pro omnibus progressionibus quorum termini sunt fractiones, harumque numeratores progressionem geometricam, denominatores vero arithmeticam constituunt. Vt vero facilius ad omnes casus accommodari possit, sumatur haec progressio,

fio,  $\frac{b}{c}, \frac{b^{i+1}}{c+e}, \frac{b^{2i+1}}{c+2e}, \frac{b^{3i+1}}{c+3e}$ , cuius terminus ge-

neralis est  $\frac{b^{(n-1)i+1}}{c+(n-1)e}$ , comparet ut hic cum illo

$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}}$  vel  $\frac{ck^{(n-1)\alpha+1}}{(c+(n-1)\alpha)a^{(n-1)\alpha}}$  erit  $\alpha$

$= \frac{e}{c}$  et  $\frac{ck^{\frac{(n-1)e}{c}+1}}{a^{\frac{(n-1)e}{c}}} = b^{(n-1)i+1}$  atque  $a =$

$\left(\frac{ck^{\frac{(n-1)e}{c}+1}}{b^{(n-1)i+1}}\right)^{\frac{c}{(n-1)e}} = \left(\frac{ck}{b}\right)^{\frac{c}{(n-1)e}} \frac{k}{b^{\frac{c}{(n-1)e}}}$ . Hic ne

$a$  pendeat ab  $n$ , debet enim  $a$  esse constans quan-  
 titas, oportet ut  $\frac{ck}{b}$  fit  $= 1$ , erit ergo  $k = \frac{b}{c}$ , at-

que  $a = \frac{b^{\frac{e-ci}{c}}}{c}$ . Quocirca terminus summatorius est

$\int \frac{b^{\frac{ne-ni}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)(e-ci)}{c}} (b^{\frac{e-ci}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} dx$ . Quae ita debet in-

tegrari ut fiat  $= 0$  si  $x = 0$ , tum vero ponere oportet  $x = \frac{b}{c}$ .

§. 9. Cognita summa progressionis indefinita habebitur summa progressionis in infinitum, si ponatur  $n = \infty$ . Terminus quidem summatorius inuentus non magis ad hunc casum quam ad aliumquemque accommodatus videtur. Est mihi vero alia methodus summas serierum infinitarum investigandi, quae latissime patet. Sit series

$\frac{b}{c} + \frac{b^{i+1}}{c+1} + \frac{b^{2i+1}}{c+2} +$

$b^{2i+1}$   
 $c + 2e$  etc. Ponatur numerus terminorum  $n$ , et summa eorum  $A$ . Augeatur numerus  $n$  unitate, augerebitur summa  $A$  termino ordine  $n+1$ , qui est  $b^{n+1}$   
 $c + ne$ . Si nunc  $n$  et  $A$  tanquam quantitates

fluentes considerentur, quia  $n$  est quasi infinites maior quam 1, erunt earum differentialia  $dn$  et  $dA$  inter se ut augmenta 1 et  $b^{n+1}$   
 $c + ne$ . Vnde pro-

dit aequatio  $dA = \frac{b^{n+1} dn}{c + ne}$ . Quae integrata dabit aequationem inter summam  $A$  et numerum terminorum  $n$ .

§. 10. Ponatur  $l(c + ne) = z$ , erit  $\frac{edn}{c + ne} = dz$  atque  $c + ne = g^z$  denotante  $g$  numerum, cuius logarithmus est 1. Est ergo  $n = \frac{g^z - c}{e}$  et  $b^{n+1}$

$= b^{\frac{g^z - c}{e} + 1} = b^{\frac{e - c}{e}} b^{\frac{g^z}{e}}$ , consequenter  $dA =$

$b^{\frac{e - c}{e}} b^{\frac{g^z}{e}} dz$ . Haec quidem aequatio ita genera-

liter instituta integrationem nisi per series non admittit. Si vero ponatur  $i = 0$ , ut prodeat seriei

$\frac{b}{c} + \frac{b}{c+e} + \frac{b}{c+2e} + \dots$  etc., habebitur aequatio  $dA$

$= \frac{b}{e} dz$  et  $A = \frac{b}{e} z + lC = \frac{b}{e} lC(c + ne)$ . Constantis quidem  $C$  non determinatur, sed tamen aequa-

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aequatio ad definiendam differentiam inter duas summas inseruit: vt sit alius numerus terminorum  $m$ , et summa  $B$  erit,  $B = \frac{b}{e} LC(c+me)$ . Ergo  $A - B = \frac{b}{e} \sqrt{\frac{c+me}{c+ne}} = \frac{b}{e} \sqrt{\frac{m}{n}}$  quia  $m$  et  $n$  sunt infinita.

§. 11. Maneat  $i=0$ , et progressio erit haec  $\frac{b}{c}, \frac{b}{c+e}, \frac{b}{c+2e}, \frac{b}{c+3e}$  etc. cuius terminus generalis est  $\frac{b}{c+(n-1)e}$ . Terminus autem summatorius est

$\int \frac{b \frac{ne}{c} - c \frac{ne}{c} x \frac{ne}{c}}{b \frac{(n-1)e}{c} (b \frac{e}{c} - c \frac{e}{c} x \frac{e}{c})} dx$ . Sumatur alia progressio

$\frac{b}{c}, \frac{b}{c+2f}, \frac{b}{c+2f}, \frac{b}{c+3f}$  etc. cuius terminus generalis est  $\frac{b}{c+(n-1)f}$  et summatorius  $\int \frac{b \frac{nf}{c} - c \frac{nf}{c} x \frac{nf}{c}}{b \frac{(n-1)f}{c} (b \frac{f}{c} - c \frac{f}{c} x \frac{f}{c})}$

$dx$ ; in quo integrato itidem ponere oportet  $x = \frac{b}{c}$ . Addantur hae duae progressionibus, scilicet terminus primus primo, secundus secundo, et ita

porro prodibit haec progressio  $\frac{2b}{c}, \frac{2bc+b(c+f)}{(c+e)(c+f)}, \frac{2bc+2b(c+f)}{(c+2e)(c+2f)}$  etc., cuius terminus generalis est  $\frac{2bc+(n-1)b(c+f)}{(c+(n-1)e)(c+(n-1)f)}$ . Terminus vero summatorius

erit  $\int dx \left( \frac{b \frac{ne}{c} - c \frac{ne}{c} x \frac{ne}{c}}{b \frac{(n-1)e}{c} (b \frac{e}{c} - c \frac{e}{c} x \frac{e}{c})} + \frac{b \frac{nf}{c} - c \frac{nf}{c} x \frac{nf}{c}}{b \frac{(n-1)f}{c} (b \frac{f}{c} - c \frac{f}{c} x \frac{f}{c})} \right)$

§. 12. Simili modo, sed vniuersalius, pro termino generali in cuius denominatore  $n$  duas repetit dimensiones, inuenitur terminus summatorius, si illius progressionis  $p$  cuplum ad huius  $q$  cuplum addatur. Obtinebitur hoc modo progressio  
Tom. V. N fio,



fio, cuius terminus generalis est  $\frac{pb}{c+(n-1)e} + \frac{qb}{c+(n-1)f}$   
 $= \frac{(p+q)bc+(n-1)b(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$ . Terminus autem summa-  
 torius huic termino generali respondens erit

$$\int \frac{p dx}{b^{\frac{(n-1)e}{c}}} \left( \frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}}} \right) + \int \frac{q dx}{b^{\frac{(n-1)f}{c}}} \left( \frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}}} \right)$$

$$= \int dx \left( \frac{pb^{\frac{n(e+f)}{c}} - pb^{\frac{n(e+f)-f}{c}} c^{\frac{f}{c}} x^{\frac{f}{c}} - pb^{\frac{nf}{c}} c^{\frac{ne}{c}} x^{\frac{ne}{c}} +}{b^{\frac{(n-1)(e+f)}{c}}}\right.$$

$$\left. \frac{pb^{\frac{(n-1)f}{c}} c^{\frac{ne+f}{c}} x^{\frac{ne+f}{c}} + qb^{\frac{n(e+f)-e}{c}} - qb^{\frac{n(e+f)-e}{c}} c^{\frac{e}{c}} x^{\frac{e}{c}}}{(b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} \right.$$

$$\left. - \frac{qb^{\frac{ne}{c}} c^{\frac{nf}{c}} x^{\frac{nf}{c}} + qb^{\frac{(n-1)e}{c}} c^{\frac{nf+e}{c}} x^{\frac{nf+e}{c}}}{(b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} \right). \text{ Ponatur } b=1$$

hoc enim modo vniuersalitati nihil decedit, erit-  
 que terminus generalis  $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$ . Sit  $cx$

$=y$ , erit  $dx = \frac{dy}{c}$ . Atque terminus summatorius ha-  
 betur  $= \int \frac{dy}{c} \left( \frac{p+q-py^{\frac{f}{c}}-qy^{\frac{e}{c}}-py^{\frac{ne}{c}}-qy^{\frac{nf}{c}}+qy^{\frac{ne+f}{c}}+qy^{\frac{nf+e}{c}}}{(1-y^{\frac{e}{c}})(1-y^{\frac{f}{c}})} \right)$

in qua formula integrata, ita vt. posito  $y=0$  ea  
 quoque fiat  $=0$ , oportet ponere  $y=1$ .

§. 13. Assumatur iam terminus generalis hic  
 $\frac{\alpha+\beta n}{\gamma+\delta n+\epsilon n^2}$ . Qui comparatus cum  $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$   
 dabit  $c = \sqrt{(\gamma+\delta+\epsilon)}$ ,  $e = \frac{\delta+2\epsilon+\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}}$ ,  $f = \frac{\delta+2\epsilon-\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}}$ ,  $p = \frac{\alpha\delta-\beta\delta+2\alpha\epsilon-2\beta\gamma+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}(\delta\delta-4\gamma\epsilon)}$  atque  
 $q = \frac{\beta\delta-\alpha\delta+2\beta\gamma-2\alpha\epsilon+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}(\delta\delta-4\gamma\epsilon)}$ . His in termino sum-  
 matorio substitutis, prodibit terminus summatorius  
 huius

huius progressionis  $\frac{\alpha + \epsilon}{\gamma + \delta + \epsilon}$ ,  $\frac{\alpha + 2\epsilon}{\gamma + 2\delta + 4\epsilon}$ ,  $\frac{\alpha + 3\epsilon}{\gamma + 3\delta + 9\epsilon}$ ,  
etc. cuius terminus generalis est  $\frac{\alpha + \epsilon n}{\gamma + \delta n + \epsilon n^2}$ .

§. 13. Eodem modo si in termino generali  
 $n$  plures duabus dimensiones habuerit, eruetur ter-  
minus summatorius, combinandis tot progressio-  
nibus simplicibus, quot dimensiones  $n$  habere de-  
bet, quemadmodum idem in casu duarum dimen-  
sionum factum est. Attamen hac ratione non  
ad quasvis, quae in huiusmodi terminis genera-  
libus contineri videntur, series perueniri potest.  
Nam quoties denominator  $\gamma + \delta n + \epsilon n^2 + \zeta n^3 + \eta n^4$   
etc. duos pluresue habet factores simplices aequa-  
les, tum progressio in tot simplices progressio-  
nes resolui nequit, neque igitur eius terminus  
summatorius inueniri.

§. 14. Hanc ob rem aliam tradam metho-  
dum, quae hos casus non excludat. Sit pro-  
gressio quaedam simplex  $\frac{1}{a}$ ,  $\frac{1}{a+b}$ ,  $\frac{1}{a+2b}$  etc. cuius  
terminus generalis est  $\frac{1}{a+(n-1)b}$ . Huius terminus  
summatorius erit  $\int \frac{1 - a^{\frac{nb}{a}} x^{\frac{nb}{a}}}{1 - a^{\frac{b}{a}} x^{\frac{b}{a}}} dx$ , vel ponatur  $ax$   
 $= y$ , erit is  $\int \frac{1 - y^{\frac{nb}{a}}}{1 - y^{\frac{b}{a}}} \frac{dy}{a}$ , in quo integrato poni  
oportet  $y=1$ . Multiplicetur hic in  $y^a dy$  et sum-  
ma huius facti  $\int y^a dy \int \frac{dy}{a} \left( \frac{1 - y^{\frac{nb}{a}}}{1 - y^{\frac{b}{a}}} \right)$  erit secundum  
N 2 modum

modum descriptum tractata terminus summatorius huius progressionis  $\frac{a}{a \cdot 6a}$ ,  $\frac{a}{(a+b)(6a+b)}$ ,  $\frac{a}{(a+2b)(6a+2b)}$  etc. posito breuitatis ergo  $\xi$  loco  $a+2$ . Huius progressionis terminus generalis est  $\frac{a}{(a+(n-1)b)(6a+(n-1)b)}$

$$\text{vel } \frac{a}{b^2 n^2 + (ab + 6ab - 2bb)n + (a-b)(6a-b)}$$

§. 15. Assumamus progressionem generalem huius generis, quae facilius ad casus quousvis adaptatur, sit eius terminus generalis

$$\frac{1}{a + (n-1)b + \frac{(n-1)(n-2)}{2}c} \quad \text{Hic cum illo termino generali comparatus dabit } a = \frac{(2b-c)^2 - 4ac + (2b-c)\sqrt{(2b-c)^2 - 8ac}}{2}$$

$$b = \frac{2b-c + \sqrt{(2b-c)^2 - 8ac}}{4}, \quad \xi = \frac{2b-c - \sqrt{(2b-c)^2 - 8ac}}{2} \quad \text{Hi}$$

valores si substituantur loco  $a, b,$  et  $a,$  (est ve-

ro  $a = \xi - 2$ ) in  $\int y^a dy \int \frac{dy}{1+a} \left( \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$  prodibit ter-

minus summatorius progressionis propositae,  $\frac{1}{a},$   
 $\frac{1}{a+b}, \frac{1}{a+2b+c}, \frac{1}{a+3b+3c}$  etc.

§. 16 Hoc modo ulterius progredi licet;

multiplicetur  $\int \frac{dy}{a} \left( \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$  in  $y^{a-2} dy$ , et facti

integrale  $\int y^{a-2} dy \int \frac{dy}{a} \left( \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$  denuo in  $y^{\xi-a-1}$

huiusque producti integrale  $\int y^{\xi-a-1} dy \int y^{a-2} dy \int \frac{dy}{a}$   
 $(1-y$

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$\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)^{\frac{a}{b}}$  erit terminus summatorius progressio-  
 nis huius  $\frac{a}{a \cdot a \cdot a} \cdot \frac{a^2}{(a+b)(a+b)(a+b)} \cdot \frac{a^2}{(a+2b)(a+2b)(a+2b)}$   
 etc. cuius terminus generalis est  $a^2 : (a+(n-1)b)$   
 $(aa+(n-1)b)(aa+(n-1)b)$  similiter  $\int y^{a-1} dy \int y^{a-1}$   
 $\int y^{a-2} dy \int \frac{dy}{a} \left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)^{\frac{a}{b}}$  est terminus summatorius  
 progressionis, cuius terminus generalis est  
 $\frac{a^2}{(a+(n-1)b)(aa+(n-1)b)(aa+(n-1)b)(aa+(n-1)b)}$ . Hoc igitur  
 modo ad omnes progressionem pervenitur, qua-  
 rum termini sunt fractiones, numeratoribus exi-  
 stentibus numeris constantibus, denominatoribus  
 autem constituentibus quamcunque progressionem  
 algebraicam.

§. 17. Si summae huiusmodi progressionum  
 in infinitum continuatarum desiderantur, oportet  
 ponere  $n = \infty$ . Hoc posito postremum cu-  
 iusque termini summatorii membrum scilicet  $\int \frac{dy}{a}$   
 $\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)^{\frac{a}{b}}$  transmutabitur in hoc  $\int \frac{dy}{a(1-y^{\frac{b}{a}})}$ . Quia  
 enim  $y$  semper est minus quam 1, praeter ca-  
 sum ultimum, quo fit  $y = 1$ , evanescet  $y^{\frac{nb}{a}}$  prae  
 1: atque ideo  $1-y^{\frac{nb}{a}}$  abibit in 1. Propterea hu-  
 ius seriei  $\frac{a}{a \cdot a} + \frac{a}{(a+b)(a+b)} + \frac{a}{(a+2b)(a+2b)} + \text{etc.}$   
 in infinitum summa erit  $\int y^{a-2} dy \int \frac{dy}{a(1-y^{\frac{b}{a}})}$ , et

huius  $\frac{a^2}{a \cdot a \cdot 6a} + \frac{a^2}{(a+b)(aa+c)(3a+c)} + \frac{a^2}{(a+2b)(aa+2b)(3a+2b)}$   
 + etc., summa erit  $\int y^{5-a} y^2 dy \int \frac{dy}{a(1-y\frac{b}{a})}$   
 et ita de reliquis omnibus.

§. 18. Sit  $b=a$ , vt fiat  $\frac{b}{a}=1$ , erit  $\int \frac{dy}{a(1-y)} = A - \frac{1}{a} l(1-y)$ . Quia posito  $y=0$  totum integrale fieri debet  $=0$ , erit  $A=0$ , adeoque  $\int \frac{dy}{a(1-y)} = -\frac{1}{a} l(1-y)$ . Multiplicetur hoc in  $y^{a-2} dy$  habebitur  $-\frac{y^{a-2} dy}{a} l(1-y)$ . Huius integrale vt inueniatur ponatur  $1-y=z$ , erit  $y=1-z$ , habebitur igitur integrandum  $\frac{(1-z)^{a-2} dz}{a} l z = (1 - \frac{a-2}{1} z + \frac{(a-2)(a-3)}{1 \cdot 2} z^2 - \frac{(a-2)(a-3)(a-4)}{1 \cdot 2 \cdot 3} z^3 + \text{etc.}) \frac{dz}{a} l z$ . Quia vero  $z^\eta dz l z = C - \frac{z^{\eta+1}}{(\eta+1)^2} + \frac{z^{\eta+1} l z}{\eta+1}$ , erit illius integrale haec series  $\frac{1}{a} (C - z + z l z + \frac{(a-2)}{1 \cdot 4} z^2 - \frac{(a-2)}{1 \cdot 2} z^2 l z - \frac{(a-2)(a-3)}{1 \cdot 2 \cdot 9} z^3 + \frac{(a-2)(a-3)}{1 \cdot 2 \cdot 3} z^3 l z + \text{etc.})$ . Hoc integrale si fiat  $y=0$  seu  $z=1$  debet fieri  $=0$ , hanc ob rem erit  $C = 1 - \frac{(a-2)}{1 \cdot 4} + \frac{(a-2)(a-3)}{1 \cdot 2 \cdot 9} - \frac{(a-2)(a-3)(a-4)}{1 \cdot 2 \cdot 3 \cdot 16} \text{ etc.}$

§. 19. Perspicuum est ex hoc integrali, quoties  $a$  sit numerus integer vnitatis maior, tum semper integralis eius terminorum numerum fore finitum, atque ideo summam progressionis definire. Attamen etiamsi terminorum numerus sit infi-

ininitus, summa propositae seriei dabitur per aliam seriem infinitam quae vero plerumque magis conuergit quam proposita, atque ideo perquam est utilis ad summam determinandam.

§. 20. Sit summa progressionis in infinitum continuatae  $\int \frac{-y^{\alpha-2}}{a} dy l(1-y)$ , quia hic est posi-

tum  $b=a$ , erit progressio ipsa  $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+2)a} + \frac{1}{4(\alpha+3)a}$  etc. Huius summa habetur si in illo integrali ponitur  $y=1$ , sed facto  $y=1-z$  est integrale illud  $\frac{1}{\alpha} (1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \text{etc.} - z + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 + \text{etc.} + z/z - \frac{(\alpha-2)}{1.2} z^2/z + \frac{(\alpha-2)(\alpha-3)}{1.2.3} z^3/z - \text{etc.})$ . Si iam fiat  $y=1$  vel  $z=0$  erit summa seriei  $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+2)a}$  etc. aequalis summae huius seriei  $\frac{1}{\alpha} - \frac{(\alpha-2)}{1.4.a} + \frac{(\alpha-2)(\alpha-3)}{1.2.9.a}$  etc. vel summa huius  $\frac{1}{\alpha} + \frac{1}{2(\alpha+1)} + \frac{1}{3(\alpha+2)}$  etc. aequalis summae huius  $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \text{etc.}$

§. 21. Praeterea alium habeo modum series valde conuergentes inueniendi, quarum summa aequalis fit seriei propositae.  $\int -y^{\alpha-2} dy l(1-y)$  aequatur ita integratum vt fiat  $=0$  si  $y=0$  huic seriei  $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16} + \text{etc.} - z + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 + \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16} z^4 - \text{etc.} + z/z - \frac{(\alpha-2)}{1.2} z^2/z + \frac{(\alpha-2)(\alpha-3)}{1.2.3} z^3/z - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.4} z^4/z + \text{etc.}$  existente  $z=1-y$ ; sed cum sit  $-l(1-y)=y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$  etc. erit  $\int -y^{\alpha-2} dy l(1-y) =$

$$= \frac{y^a}{a} + \frac{y^{a+1}}{2(a+1)} + \frac{y^{a+2}}{3(a+2)} \text{ etc.}$$
 Haec series si  $a$  est numerus affirmativus est aequalis illi quicquid sit  $y$ , et ita multis modis series eiusdem summae reperiuntur, quarum altera alterius ope facilius summatur.

§. 22. Exemplo rem illustrabo fit  $a=1$ , habebitur trium sequentium serierum summa aequalis huic

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \text{ etc.}$$
  

$$-z - \frac{1}{4}zz - \frac{1}{9}z^3 - \frac{1}{16}z^4 \text{ etc.}$$
  

$$+ \frac{y}{z} + \frac{y^2}{4} + \frac{y^3}{9} + \text{etc.}$$
  

$$+ z/z + \frac{1}{2}z^2/z + \frac{1}{3}z^3/z \text{ etc.}$$
  
 Quia vero est  $z + \frac{1}{2}zz + \frac{1}{3}z^3 \text{ etc.} = 1/(1-z^2) = 1/y$  erit  $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{y+z}{1} + \frac{y^2+z^2}{4} + \frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} + \text{etc.} + y/z$ , est hic  $y+z=1$ , et manifestum est tales loco  $y$  vel  $z$  numeros assumi posse, ut series maxime convergat. Id vero evenit quando  $y=z$  vel utrumque  $= \frac{1}{2}$ , eritque hoc casu  $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = 1 + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \frac{1}{400} + \text{etc.} = \square/2$ . Hoc modo summa progressionis  $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$  valde prope haberi potest, est enim  $\square/2 = 1.2 + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} \text{ etc.}$  Summa progressionis  $1 + \frac{1}{8} + \frac{1}{3.8} \text{ etc.}$  est quam proxime  $= 1$ , 164481 et  $\square/2 = 0$ , 480453, ergo summa seriei  $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$  est  $= 1$ , 644934 q.p. Si quis autem huius seriei summam addendis aliquot terminis initialibus determinare voluerit, plusquam mille terminos

nos

nos addere deberet, quo nostrum inuentum numerum reperiret.

§. 23. Ex his igitur methodum percipere licet, quomodo cuiuslibet progressionis, cuius termini sunt fractiones, quarum denominatores constituunt progressionem quamcunque algebraicam, terminum summatorium inueniri oporteat. Equidem, ut hic rem considerauimus, numeratores deberent esse quantitates constantes; sed non difficulter haec methodus extendetur ad eas quoque progressionem, in quibus numeratores progressionem etiam quamcunque algebraicam faciunt. Propterea haec methodus ad omnes progressionem quarum termini generales algebraice possunt exponi, accommodari potest, eiusque opetermini summatorii inueniri. Excipiendi tamen sunt casus, quibus terminus generalis irrationalis est.